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Alternatives in Quantum Theory

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Professor Feynman,

This is not for review; just for your amusement. As a former student of yours at Tech in the 50's I thought this interpretation might tickle you. I'm going to submit this to the Am. J. Phys. when I get the figures neatified.

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Abstract. *Quantum mechanical descriptions in terms of momentum and position are identified as alternatives under the condition of equal complex-valued Lebesgue square integrability. While this does not change any of the formal results obtained in quantum mechanics, it does shed a different interpretive light on the steps that lead up to these results. Instead of being independent, even in concept, momentum and position are identified as being the same thing, merely seen from different views. Neither is required to complement the descriptive capability of the other, since each forms a complete alternative in its own right. Apparent complementarity, as well as mutual indeterminacy of codescription, comes about whenever an attempt is made to overlap these two descriptions to form a third codescription which tries to maintain both the terminology and the net integration values of the separate alternatives. Noncommutation of associated operators is a direct consequence of the alternative nature of momentum and position, and the local to global character of the particular map between these alternatives leads to an apparent point-wave duality in their descriptions. It is suggested that challenges to the contemporary interpretation of quantum theory should begin with the assumptions that lead to its formulation involving alternative representations.*

INTRODUCTION

In the early development of quantum mechanics, a great deal of concern was expressed over the failure of certain matrices and operators to commute.[1,2,3] In an attempt to understand the significance of this noncommutativity, Heisenberg was led to an analysis which disclosed a relationship of mutual indeterminacy between them.[4] Subsequently known as the Uncertainty Principle, this failure of coprecision has been a continuing source of great anguish in both practical science and philosophical thinking.

The mathematical expression of the Uncertainty Principle would seem to lie at the heart of this anguish. Heisenberg suggested that this expression arose because of the inescapable disturbance which any observation must cause to that which is observed, and that it established the mathematical limits of coprecision caused by observation of such complementary properties.[4,5]

Even before Heisenberg developed this famous relation, Bohr was concerned about the inextricable involvement of certain complementary properties in physical description, such as particles and waves. An entity could apparently be wavelike under one set of circumstance, and particlelike under another. Each complemented the other, yet both seemed necessary for a complete description of observed reality. To Bohr, the uncertainty relations not only provided a mathematical justification for the known complementary properties, but seemed to introduce complementarity as a fundamental concept that was intimately involved with the process of observation.

The inevitable consequence of interpreting the uncertainty relations as due to the process of observation, and through it complementarity of noncommuting descriptors, leads to certain difficulties with causation and the influence of the observer. The literature is rich in discussions of this matter.[6,7,8,9,10] Furthermore, every effort at dislodging these disturbing consequences is directed either at finding a flaw in the uncertainty relations themselves, or suggesting that they represent an incompleteness in descriptions of quantum phenomena. Yet, the uncertainty relations remain unshaken even under the most vigorous onslaught.

My own work on quite another subject, analysis of the problem of reconciling subjective perception of sound with objective measures of the ingredients of that perception, has led me to a geometric structure which parallels the interpretive aspects of quantum mechanics. In this correspondence, I will describe one aspect of my work as it relates to the interpretation of quantum theory and describe what I believe to be a reasonable explanation of what, up to now, has seemed mysterious in that theory. This interpretation leaves the formalism of quantum mechanics, including the uncertainty relations and everything that follows from them, completely intact. But by directing attention to the previously unquestioned procedures which led up to the uncertainty relations, sheds a different interpretive light on them and on the noncommutation of observables.

BACKGROUND

There are several key steps in the development of quantum mechanics which should be considered, not to question their validity, but for the support they provide for the concept which I wish to introduce in this correspondence.

It was recognized very early in the development of quantum theory that certain pairs of entities did not commute in the order of their application. This appears not only in the matrix formulation of Heisenberg, but as an operator notation in

Schroedinger's wave formulation and as special numbers in Dirac's theory. An excellent presentation of the early development of this noncommutating relation can be found in Van der Waerden.[11] Nonvanishing of the commutator of certain pairs of operators or of their equivalent matrices was clearly identified with quantum behavior, since this is not a characteristic of their classic physics counterparts. Contemporary discussions of the significance of the commutators of operators goes far beyond the realm of quantum mechanics,[12] but still does not attach any particular significance to the relationship between noncommutating pairs of operators.

Investigating the question of whether or not the formal structure of quantum mechanics allowed for the determination of the properties signified by such noncommutating enties, Heisenberg resorted to the Dirac-Jordan transformation theory.[4,6] This theory relates the description of position and momentum through the Fourier transform. The state function in terms of the position coordinate is the Fourier transform of the state function in terms of the momentum coordinate. Heisenberg assumed a Gaussian position distribution and obtained a Gaussian momentum distribution. He noted that there is an inverse spread in these distributions such that their product is a constant. It was immediately obvious that a state function could not be indefinitely narrowly distributed in both position and momentum. In order to understand why momentum and position could not be codetermined with indefinite precision, Heisenberg proposed the now famous "gamma ray microscope" thought experiment. No attempt was made to question the steps which gave the result; instead, it was the result itself which was scrutinized. In attempting to explain the result that came from the Fourier transform relation, the thought experiment suggested that it was a consequence of a disturbance in momentum caused by the attempt to observe position. Although there were subsequent epistemological modifications to this interpretation,[13,14] the mutual indeterminacy which these results implied still remains associated with the process of observation. Subsequent derivations of this mutual indeterminacy showed it to be directly related to the commutator of Hermitian operators.[15] Thus, mutual indeterminacy of pairs of descriptors is intimately linked with their commutator.

Another indication of the mathematical link between position and momentum is available in the formal procedure used in the wave function formulation. The quantum mechanical representation of the cartesian component of linear momentum can be clearly identified as the Fourier transform of the associated cartesian coordinate.[16] That, however, is not the identification traditionally presented to students of the theory. Instead, the associated operator notations are presented as rules to be used without comment.

Another indicator which, I contend, supports the concept I wish to introduce is provided by the apparent mutual relationship which exists between certain pairs of descriptive entities. In classic physics, the dual role played between a point (or particle, or corpuscular) description and the wave description is evident as far back as Newton's corpuscular theory of light and Huyghen's wave theory of light. Fermat's Principle of Least Time derives from the undulatory theory, while Maupertuis' Principle of Least Action was discovered through investigations on Newton's corpuscular theory of light.[17]

This dual nature of light played a significant part in the development of the early quantum theory. De Broglie showed a wave aspect of matter, and Schroedinger's formulation of the theory has clear roots in classical wave theory.[18] Heisenberg's famous thought experiment, which he formulated with regard to the indeterminacy relations, demonstrated an inextricable involvement of position and momentum with the point-wave aspects of the observation process.

Bohr considered such mutual behavior to be in the true nature of things as expressed by his concept of complementarity. Precisely what complementarity means is somewhat open to question, since Bohr was less than explicit in its description, but there are a number of excellent discussions on this subject.[6,14,19] A general interpretation of complementarity is that a complete description of a physical process requires pairs of entities which have the intrinsic property that any attempt to define one of them to increasing certainty must lead to a corresponding uncertainty in the other.

There are thus three indicators of the interdependence of the quantum mechanics entities which are known as momentum and position: their nonvanishing commutator, the Fourier transform notation of their representations, and the apparent complementary nature of their description. I contend that they all imply the same consequence: momentum and position are the same thing, seen through different frames of reference. They are alternative representations of each other; a contention I now hope to show.

ALTERNATIVES

In dealing with what were obviously different ways of describing the same phenomenon, my own work led me to consider that there could always be alternative descriptions. If one observer has a particular description of a process or event, then that description could never be privileged in the sense that there can be no other possible way of describing the same thing. It should, in principle, be possible to map that observer's frame of reference, with all its structural rules, into alternative frames of reference, either of the same dimensional basis or, in the case of sound perception, into bases of different dimensionality. This I called a Principle of Alternatives.[20] Observers in each of these alternatives should see the same event acting as if it belonged solely to that observer's frame of reference. This concept identifies transformation as a process of mapping between functional alternatives under a defined set of conditions. If there is a valid functional representation f which is expressed in a frame of reference x, then there will exist maps m which can transform this f(x) into equally valid representations in terms of alternative frames of reference. These representations, which are equally valid under a defined set of conditions C, will be said to constitute a set of alternatives under conditions C.

In the case of linear signal analysis, as well as quantum mechanics, the condition of complex-valued Lebesgue square summability, in which the representations are of class L^2, defines a particularly useful set of alternatives.[21] This can be used to represent either conservation of energy, or conservation of probability in the descriptions. Since total content, expressed through L^2, is the only conserved property, such alternatives may exist at different dimensionalities and different units of expression.

If the geometry of representation is Euclidean, then such functional representations define a Hilbert space.[22] Since an N-dimensional Riemannian representation can be mapped to a Euclidean representation of 2N(N + 1)-dimensions,[20,23] these alternatives suffice to represent both classes of geometry. The terms transformation, mapping, function and operator are often used in the same context. In this correspondence, I shall use them to mean a procedure by which a defined entity under one set of representation may be converted to another representational form.

THE FOURIER TRANSFORM

Let us begin by considering the Fourier transform as it defines a map between functional alternatives. In N-dimensional space, the complex-valued Fourier transform has the expression,

$$g(\sigma) = (1/a)^{1/N} \int_s \exp\{i<\sigma,s>\} f(s)\, ds, \tag{1}$$

where $<\sigma,s>$ is the zero curvature hyperplane, expressed through the inner product relation,

$$<\sigma,s> = \sigma_1 s_1 + \sigma_2 s_2 + \cdots + \sigma_N s_N, \tag{2}$$

a is a normalizing constant, ds is the Lebesgue measure and integration is taken over the whole of space s. This is a map of the form,[20]

$$g(\xi) = \int_x m(\xi,x)\, f(x)\, dx, \tag{3}$$

where $\xi = \xi_1, \xi_2, \cdots \xi_M$ and $x = x_1, x_2, \cdots x_N$. This transforms an expression f, in frame of reference x, into an alternative expression g in a different frame of reference ξ. The mapping kernel $m(\xi,x)$ expresses certain parameters ξ in terms of frame of reference x. The product of this mapping kernel with any measurable f(x) results in a distribution over the whole of space x whose net Lebesgue sum over x leaves a representation in terms of parameters ξ.

If f is of class L^2,

$$\int_x |f(x)|^2\, dx < \infty, \tag{4}$$

and if mapping kernel $m(\xi,x)$ preserves total Lebesgue measure, then g is also of class L^2, and

$$\int_\xi |g(\xi)|^2\, d\xi = \int_x |f(x)|^2\, dx. \tag{5}$$

In that case, the transform (3) is an isomorphism of L^2 onto L^2 which preserves total measure and maps space x, of dimension N, onto space ξ of dimension M, where N may be the same or different than M.

The mapping kernel identifies the way in which coordinates ξ will appear in space x. The parameters ξ become the coordinates of the transformed expression. Representations $g(\xi)$ and $f(x)$ are L^2 alternatives under the map $m(\xi,x)$.

In this conception, the Fourier transform can be identified as a particular map which joins two classes of alternative. In geometric terms, the Fourier transform is a map between alternatives of the same dimensionality. The Fourier transform provides an alternative view of the same entity in the same number of dimensions but in a different frame of reference. Its mapping kernal causes a pairwise coordinate alignment such that functional dependence along the i^{th} coordinate in s, s_i, is mapped to functional dependence only along the i^{th} coordinate in σ, σ_i, where alternative coordinates s and σ have inverse units of expression.

The Fourier transform on L^2 is a one-to-one norm preserving map (Parseval's equation) of L^2 onto L^2. It preserves inner products, thus is unitary. In this regard a Fourier transform defines an isomorphism of the Hilbert space onto itself. By virtue of this property, the Fourier transform preserves geometry and maps a Euclidean geometric structure in s into a Euclidean geometric structure in σ, and conversely.

INDETERMINACY DUE TO COMBINING ALTERNATIVES

Most of us get so involved in the machinery by which mathematical expressions are manipulated that we tend to overlook the meaning of those expressions. When an expression involves an equal sign, we must recognize what this sign means. An equal sign clearly separates two entities which are to be interpreted as equivalent to each other under some agreed set of conditions. In the equation which identifies the Fourier transform, the representation $g(\sigma)$ must be the same thing as the expression on the opposite side of the equal sign. This is so fundamental that it should require no comment, but failure to recognize its significance has, in my opinion, led to a tragic error of interpretation in physics.

The equal sign clearly signifies that the entity $f(s)$ under the procedure

$$(1/a)^{1/N} \int_s \exp\{i<\sigma,s>\}(\cdot)\ ds \qquad (6)$$

is equal to the entity $g(\sigma)$ everywhere except over sets of Lebesgue measure zero. This means that if $f(s)$ describes something which has Lebesgue measure and is of class L^2, then $g(\sigma)$ is an alternative description of the same thing with the same total L^2 summed measure. The frame of reference s and the frame of reference σ are not, and cannot be, independent since each is precisely mappable into the other under the rule which the equation sets out. The fact that there is an equation should supply the clue that these entities and their frames of reference could never be independent.

The Fourier procedure identifies not only the way in which each elemental point in s appears as a description in σ, but the tradeoff between regions of confinement of the same L^2 summation in s and σ. A simple manipulation shows that the hyperplane kernel causes a mutual inverse spreading between the region of confinement of measure for a particular description along s_i and the corresponding region of confinement for the alternative description along σ_i For a clustered distribution, such as the Gaussian, the Fourier procedure results in the well-known relation,

$$\Delta s_i \, \Delta \sigma_j \geqq \tfrac{1}{2} \, \delta_{ij} \,, \qquad (7)$$

where δ_{ij} is the Kronecker delta, which is unity if the indices i and j are alike and zero otherwise, and Δ stands for the effective width of the region of confinement of the net L^2 summation over the range of the whole coordinate. It is clear that the Gaussian distribution leads to the smallest product of Δ's. Other distributions will lead to products of mutual spreading which are larger than that for the Gaussian. Thus the inference that the regions of spreading are inversely related through a fixed constant is not strictly correct for clustered distributions in general.

A representation in s and a representation in σ are descriptions of the same thing as seen from these different frames of reference. That part of the representation confined to region Δs_i in frame of reference s is confined to region $\Delta \sigma_i$ in alternative frame of reference σ. The narrower one makes Δs_i, the broader becomes $\Delta \sigma_i$, in conformity with relation (7). If one attempts to form a third description in terms of both s and σ, then that part of this joint description whose L^2 sum is confined to Δs_i must also be confined to $\Delta \sigma_i$ if the net L^2 sum of this codescription is to be maintained at the same value as that for the separate descriptions. In the large, where descriptive variations are coarse, it might appear that s and σ can be independent attributes in this third codescription. But as one refines the description, the true alternative nature of s and σ will make itself felt as an irreducible indeterminacy of codescriptions in L^2. In quantum mechanics, the squared magnitude of the wave function is interpreted as probability, whose net L^2 sum must be unity whether for single or codescriptive applications. The associated quantum mechanical mapping kernel involves the hyperplane $\langle p,q \rangle$ divided by the constant factor $h/2\pi$. The Heisenberg indeterminacy relations of quantum mechanics can thus be directly identified with this process, and the limiting indeterminacy in p and q becomes $\frac{1}{2}(h/2\pi)\delta_{ij}$, as it must be. The simplified notation $\hbar$ is not used here in order to bring out the role of Planck's constant as a normalizing coefficient in a hyperplane angle relationship with p and q.

The hyperplane kernel also establishes the relation that each elemental point s_o in s is mapped to appear as a corresponding distribution

$$\exp\{i\langle\sigma, s_o\rangle\} \tag{8}$$

in the alternative frame of reference. This distribution is a wave. There is a point-wave alternative relationship between coordinates s and σ, and on any representations expressed in these coordinates. Attempting to form a third description by combining both s and σ will lead to an interpretation that this third description has both a placelike property and a wavelike property which coexist in a most unusual manner. Each will seem necessary in order to form a complete description (in terms of both s and σ), yet attempts to obtain greater accuracy in the determination of either of them seems to result in a loss of precision in the other. Furthermore, interpretations of the same process can either be more placelike or more wavelike, depending upon the choice of coordinate emphasis.

Indeterminacy, apparent complementarity and point-wave duality result from attempting to combine alternative representations into a common overlap view. These things are the penalty that must be paid in order to maintain both the terminology and the net L^2 value of the separate alternatives. We will next see that alternative operational procedures also impose constraints on their use when applied in a common frame of reference. In particular, the famous noncommutation of certain operators is a consequence of their alternative relationship.

NONCOMMUTING OF OPERATORS

The general mapping procedure establishes the way in which an operation R in a frame of reference x will appear as an equivalent operation S in the alternative frame of reference ξ. If a representation f is mapped to an alternative representation g under the map m, then, in operator notation,

$$m[f] = g. \qquad (9)$$

In order to specify that g is a valid alternative to f, and in particular that there is nothing contained in f that does not appear in g, there must exist another map n by which f can be recovered from g everywhere (except over sets of measure zero). That is,

$$n[g] = nm[f] = f. \qquad (10)$$

Since successive maps m and n recover the original form,

$$nm = mn = I. \qquad (11)$$

The relationships are symbolized in the mapping diagram of Figure 1.

If there is an operation R in the space of f which corresponds to an operation S in the m-transform space of g, then there are two ways in which an f may be carried into a form S[g], namely,

$$mR[f] = S[g] = Sm[f]. \qquad (12)$$

An operator equivalence relation on any measurable f is thus,

$$mR = Sm. \qquad (13)$$

This can be cast into the following equivalence relationships for alternative operators:

$$R = nSm$$
$$S = mRn. \quad (14)$$

These relationships can be readily verified for the case where m is the Fourier transform and R and S are transform equivalent operations, such as,

$$R = \partial/\partial s_i \quad ; \quad S = i\sigma_i. \quad (15)$$

Relation (14) is an operational statement that S is the m-transform equivalent of R, while R is the n-transform equivalent of S, as defined under relation (11). Operational procedures R and S may be applied in the same frame of reference. The question to be answered is: does S followed by R yield the same result as R followed by S, when S and R are related to each other by (14)? The answer is, no, the two sequences do not give the same result. R and S do not commute. The underlying reason for this failure to commute lies in the difference between the procedures of operation and transformation between alternative forms. An operation produces a different version of the operand, which is expressed in the same frame of reference; whereas, transformation between alternative forms is a map to a distinctly different frame of reference. Any mapping diagram, such as Figure 1, must take this into account. One can, in some measure, use the same coordinate basis to compare the initial and final versions of an operational procedure, while there is no way to compare transform alternatives in a common frame of reference. That is, in fact, the reason for the indeterminacy of form that results when attempting to combine alternatives into a common descriptive basis. Differentiation, multiplying by a monomial and convolution are examples of operations; while the Fourier transform is an example of a transformation between alternatives. Because of this difference, the procedures of transformation between alternative representations and operation do not commute. This can be seen from the mapping diagram of figure 1 where, if g is a valid alternative to f under map m, then for any R,

$$mR \neq Rm. \tag{16}$$

It will now be shown how this causes the noncommutation of R and S.

The sequential procedures of S followed by R and R followed by S can be written in terms of one operation only, say S, as follows:

$$\begin{aligned} RS[f] &= nSmS[f] \\ SR[f] &= SnSm[f]. \end{aligned} \tag{17}$$

The two procedures of relation (17) are diagrammed in Figure 2. As a trial, let us assume that R and S did commute, so that the two nodes in the upper righthand corner of Figure 2(a) coincide. If this were true, and RS did produce the same result as SR, then Figure 2(b) shows that there can be two ways by which Sm[f] could be carried into RS[f]:

$$SnSm[f] = n\hat{R}Sm[f] , \tag{18}$$

where $\hat{R}$ is the m-transform equivalent of S, as shown in Figure 1.

We can consider Sm, which is map m followed by operation S, to be an equivalent procedure P, where,

$$P = Sm. \tag{19}$$

In Figure 2(b) the lower righthand node can be reached by two paths, leading to the equivalence relation,

$$SmS = \hat{R}Sm , \tag{20}$$

or

$$PS = \hat{R}P. \tag{21}$$

But, from the definition of $\hat{R}$, S followed by m produces the same result as m followed by $\hat{R}$, as,

$$mS = \hat{R}m. \tag{22}$$

Comparing relations (19), (21) and (22) it follows that that original assumption that RS produces the same result as SR cannot be true. Hence,

$$RS - SR = [R,S] \neq 0. \tag{23}$$

The noncommutation of alternative operational procedures, when applied in the same frame of reference, results from the distinction between the procedures of mapping between alternative forms, as expressed by relations (1) and (3), and the procedure of operation. Operational procedures that do not jointly fall under the limitations of relation (14) are allowed to commute. Thus, differentiation commutes with convolution, but differentiation cannot commute with multiplication by a monomial (relation (15)), nor can convolution commute with its Fourier transform equivalent of functional multiplication. Perhaps of greatest significance is the fact that the foregoing development was presented without limitation to the type of alternative map m. Noncommutativity is not a unique property of the Fourier transform, nor does its appearance in quantum mechanics need to signify anything more than that the associated properties are alternatives as defined herein.

CONCLUSION

The concept of alternatives brings a simple geometric interpretation to the mathematical formalism which lies at the foundation of quantum mechanics. None of the formalism has been challenged or replaced; nor has any doubt been cast on the results which derive from that formalism. What this concept brings is the interpretation that the quantum mechanics properties of momentum and position are correlated for the simple reason that they both represent the same thing and, hence, are nothing more than different versions of each other. Because these particular alternatives are linked by a local to global map, combining them in a common description must result in mutual spreading which gives the appearance of complementarity whenver coprecision is attempted.

This result is in no way dependent on quantum mechanics itself but comes from an investigation of the formal nature of the mathematical symbolism which is used to model quantum mechanics entities. A complex-valued functional representation, which is of class L^2 over the whole of its particular frame of reference, can be recast into an alternative form under procedures which maintain the net L^2 summability over the whole of the new frame of reference. If total content, in the sense of net Lebesgue summation, is maintained, then these different ways of describing the same thing are alternatives under L^2. The procedure by which one such alternative may be mapped into another such alternative must preserve L^2 everywhere except over sets of Lebesgue measure zero. The Fourier transform on L^2 is such a map. Thus entities which are joined by Fourier transformation under conditions of equal square summability are L^2 alternatives. One might not necessarily begin an analysis by considering that paired functional representations are alternatives, but the moment they are expressed through an L^2 preserving map, such as the Fourier transform, they must be interpreted as such.

Concern about interpretation of the indeterminacy relations should begin with the *equations* which lead up to them, not with trying to ask what they mean as a form consisting of a resultant product of dispersions. By looking at the meaning of the equations which lead up to the indeterminacy result we also see that the celebrated noncommuting of paired operators is completely consistent with the concept that the entities which these operators represent are alternatives.

I offer these results, from quite a different field of application, for the interesting interpretation which they can bring to the physics of the microcosm. Space does not permit discussion of the possibilities here, but it must be evident that one should look backward at the procedures which led up the the formulation of momentum and position as alternatives, as well as look forward to reinterpret the philosophical implications with regard to completeness and the influence of the human observer.

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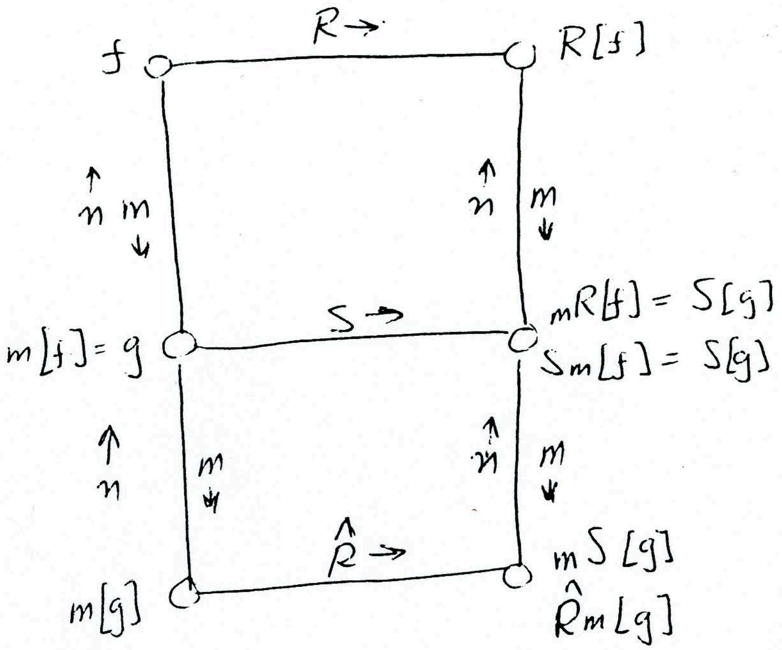
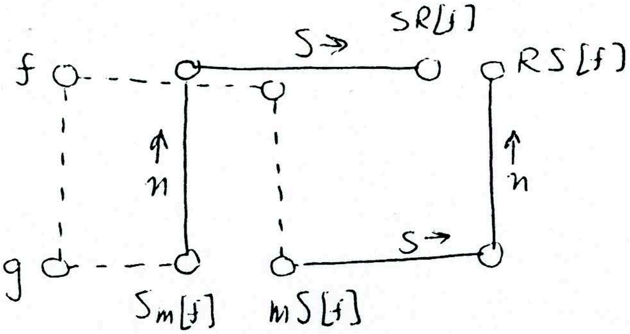


Figure 1. Mapping diagram showing the relationship between operations (capital letters and lateral displacement) which act on representational forms to create a new form within the same frame of reference; and alternative maps (lower case and vertical displacement) which transform to a new frame of reference. Each node is a representational form, with the direction of arrows indicating forward procedures; inverse procedures act contrary to the direction of arrows. Nodes at the same vertical location correspond to representational forms within the same frame of reference, while nodes atthe same horizontal location correspond to alternative representations of those above and below it. In this example, operation S is the m-transform equivalent of operation R and, in turn, $\hat{R}$ is the m-transform equivalent of S.



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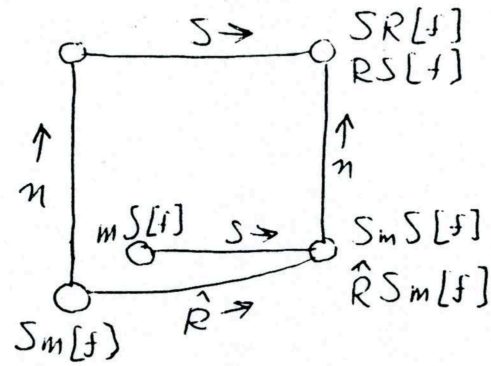

2(b)

Figure 2. Mapping diagrams for sequential operations R and S applied to a functional representation f, where S is the m-transform equivalent of R: (a) Showing paths by which the intermediate stages of mS[f] and Sm[f] are carried into RS[f] and SR[f], respectively. (b) Showing the relationship with operation $\hat{R}$ that would exist if the nodes SR[f] and RS[f] coincide; since the state of the lower righthand node would violate the definition of $\hat{R}$, the upper righthand nodes cannot coincide, with the meaning that R and S cannot commute.

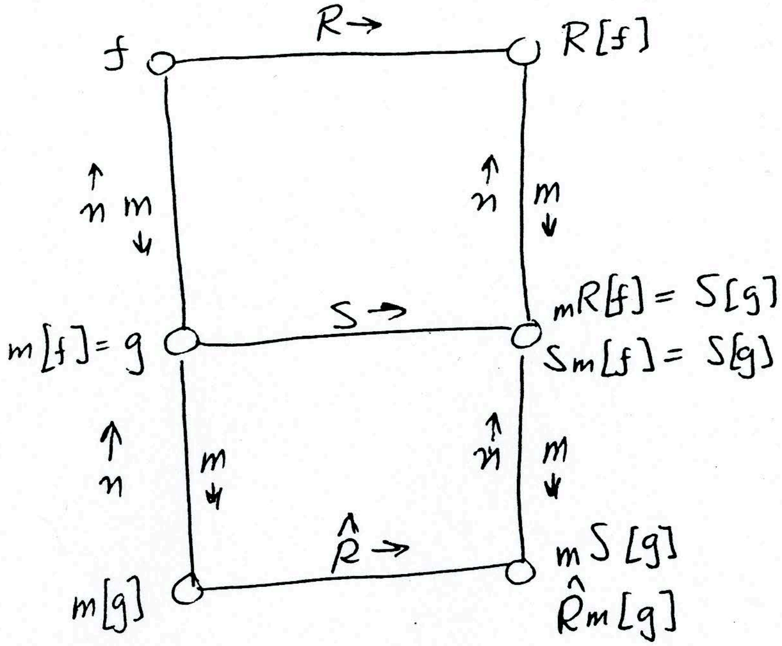

Figure 1. Mapping diagram showing the relationship between operations (capital letters and lateral displacement) which act on representational forms to create a new form within the same frame of reference; and alternative maps (lower case and vertical displacement) which transform to a new frame of reference. Each node is a representational form, with the direction of arrows indicating forward procedures; inverse procedures act contrary to the direction of arrows. Nodes at the same vertical location correspond to representational forms within the same frame of reference, while nodes atthe same horizontal location correspond to alternative representations of those above and below it. In this example, operation S is the m-transform equivalent of operation R and, in turn, $\hat{R}$ is the m-transform equivalent of S.

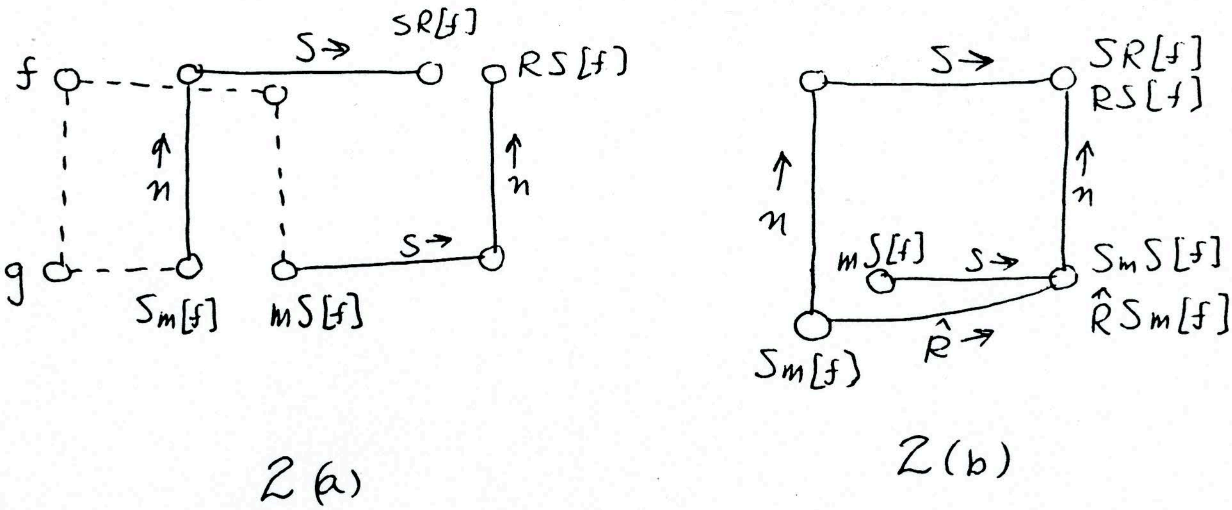

Figure 2. Mapping diagrams for sequential operations R and S applied to a functional representation f, where S is the m-transform equivalent of R: (a) Showing paths by which the intermediate stages of mS[f] and Sm[f] are carried into RS[f] and SR[f], respectively. (b) Showing the relationship with operation $\hat{R}$ that would exist if the nodes SR[f] and RS[f] coincide; since the state of the lower righthand node would violate the definition of $\hat{R}$, the upper righthand nodes cannot coincide, with the meaning that R and S cannot commute.